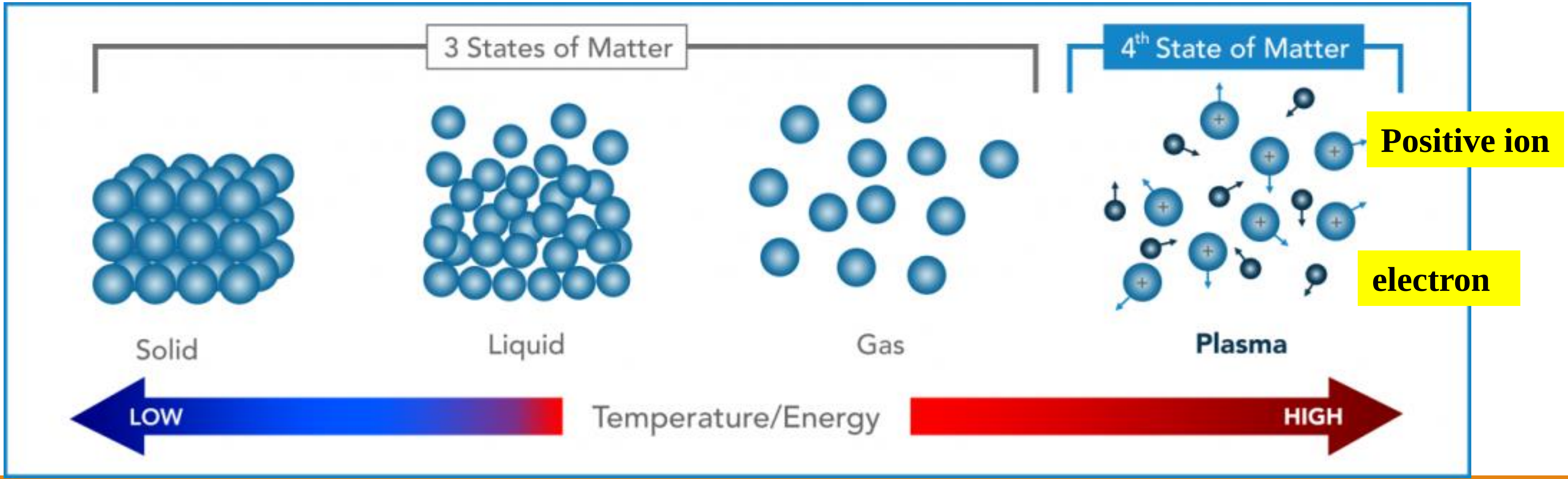
Damped simple harmonic motion

17TH AUGUST 2020

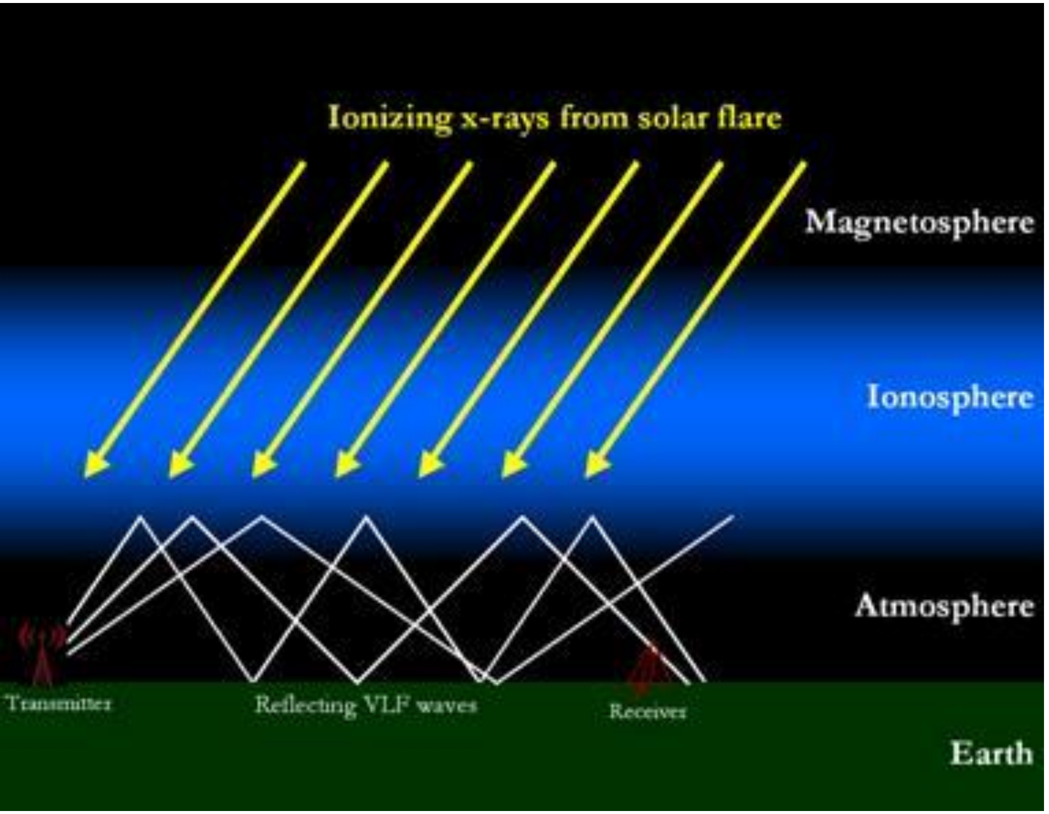
Plasma oscillations-what is plasma?

Plasma is simply an ionized or electrically charged gas, and is often described as the fourth state of matter, i.e. when energy is added to a solid (first state) it becomes a liquid (second state); with more added energy it becomes a gas (third state) and when further energy is added it eventually disassociates to become a plasma.



https://tetronics.com/our-technology/what-is-plasma/

Plasma oscillation : the ionosphere



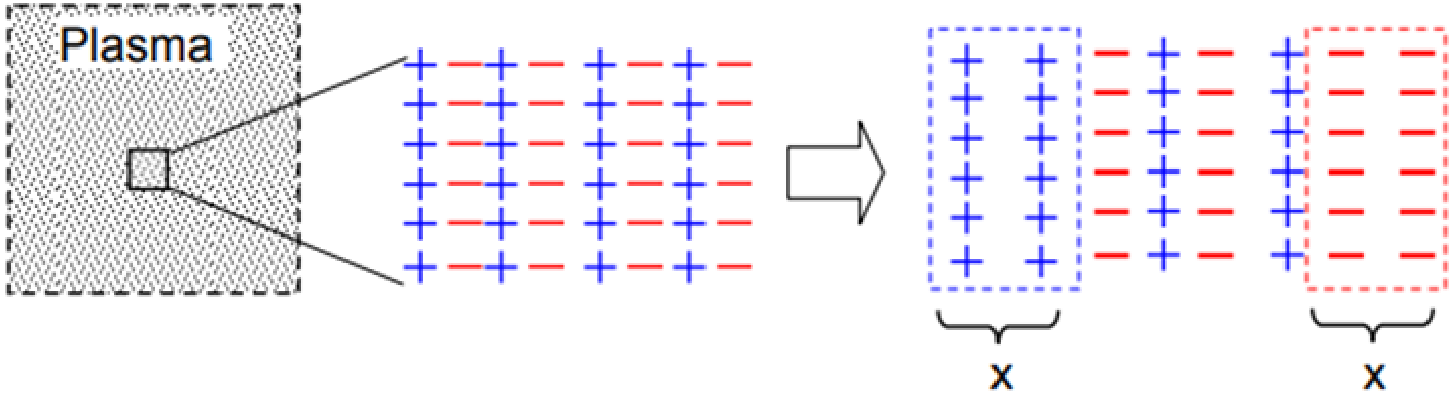
- The ionosphere is defined as the layer of the Earth's atmosphere ionized by solar and cosmic radiation. It lies 75-1000 km above the Earth.
- Because of the high energy from the Sun and from cosmic rays, the atoms in this area have been stripped of one or more of their electrons, or “ionized,” and are therefore positively charged. The ionized electrons behave as free particles. **The particles are in the plasma state.**
- The ionosphere influences radio propagation to distant places on the Earth, and between satellites and Earth.

http://solar-center.stanford.edu/SID/activities/ionosphere.html

Plasma oscillations : Plasma characteristics

- The plasma is overall neutral, i.e., the number density of the electrons and ions are the same.

The plasma volume is given as l^3.



Schematic diagram of the plasma, with the inset showing a typical volume within the plasma with equal densities of positive ions and electrons.

Small volume of plasma in which the Electrons are displaced to the right by an amount x, while the ions are fixed.

Note the number density refers an amount of particle in any volume.

Plasma oscillation : the analysis

- The positive region on the left side and the negative region on the right side can be considered as two parallel infinite charged planes with a thickness of x.

Electric field between two charge planes : $E =$

where Q is the total charge in the slabs, n_e is the number density (in m^{-3}) and e is an electron charge.

The restoring force per unit area on the electrons $F =$

The equation of motion on the electrons is written as ,

This is a standard form of simple harmonic oscillator.

The angular frequency known as **plasma frequency** is found to be $\omega_p =$

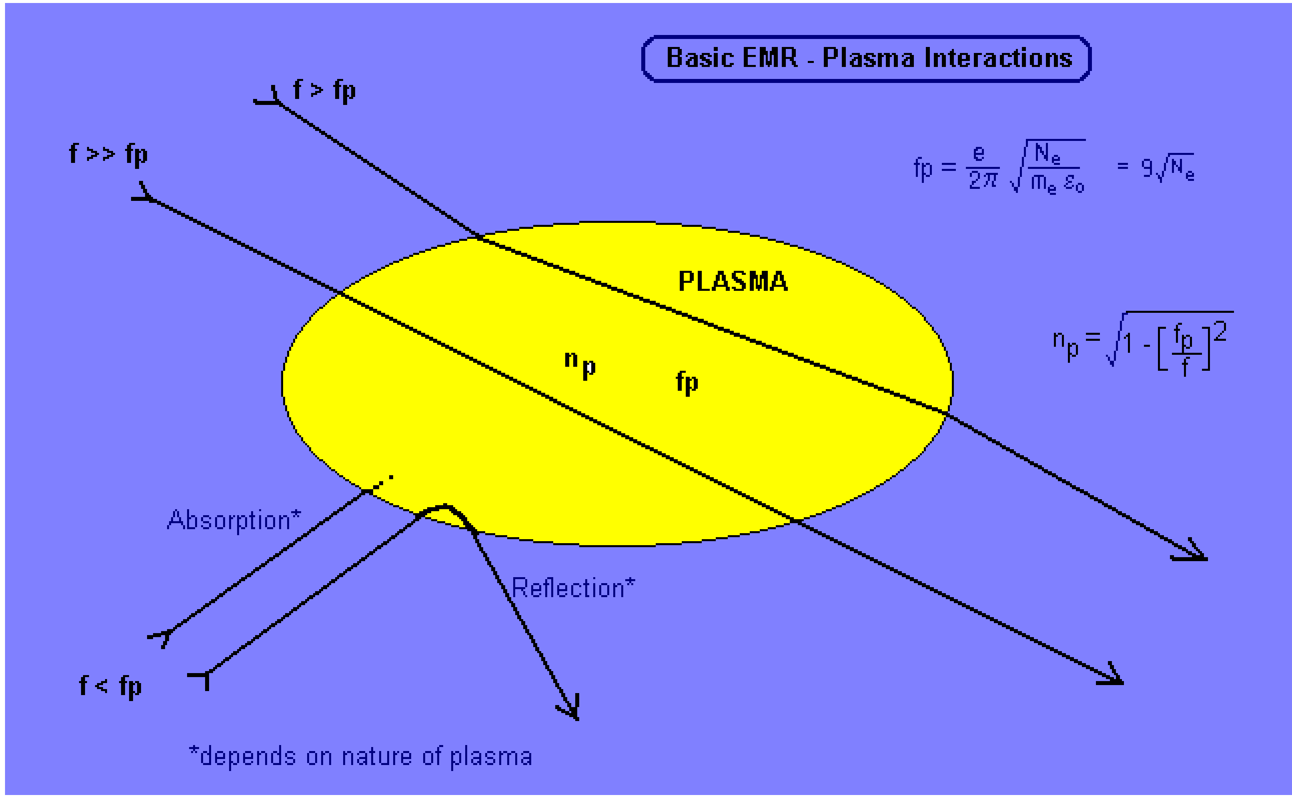
Plasma frequency and ionosphere

- The ionosphere is what we term a weak plasma, as only one percent of the neutral atoms in the upper atmosphere are ionized. Traces of ionization exist from about 80 km to 1000 km in altitude, with the peak ionization occurring around an altitude of 300 km. The maximum ionization can vary from about 10^{10} to 10^{13} electrons per cubic meter.

From https://spaceacademy.net.au/env/spwx/raiono.htm

- **Determine the range of the plasma frequency!**

- Note : Only radio waves with frequencies significantly higher than this plasma frequency can propagate through the plasma. The lower frequency can bounce off the ionosphere.



Phase space

- In classical mechanics, the phase space is the space of all possible states of a system; the state of a mechanical system is defined by the constituent positions and velocities or momenta.
- The horizontal coordinate represents the position x while the vertical coordinate represents the velocity v.
- The future state of motion of such a particle is completely specified if its positon and velocity are known simultaneously.
- The trajectory of these points in phase space represents the complete time history of the particle.

Simple harmonic oscillator : no damping force

- A general solution for a simple harmonic motion such as a simple pendulum may be written as $x(t) = A\sin(\omega_0 t + \phi_0)$
- The corresponding velocity is found to be $\dot{x}(t) = A\omega_0 \cos(\omega_0 t + \phi_0)$
- The trajectory of the oscillator in the phase space becomes

----

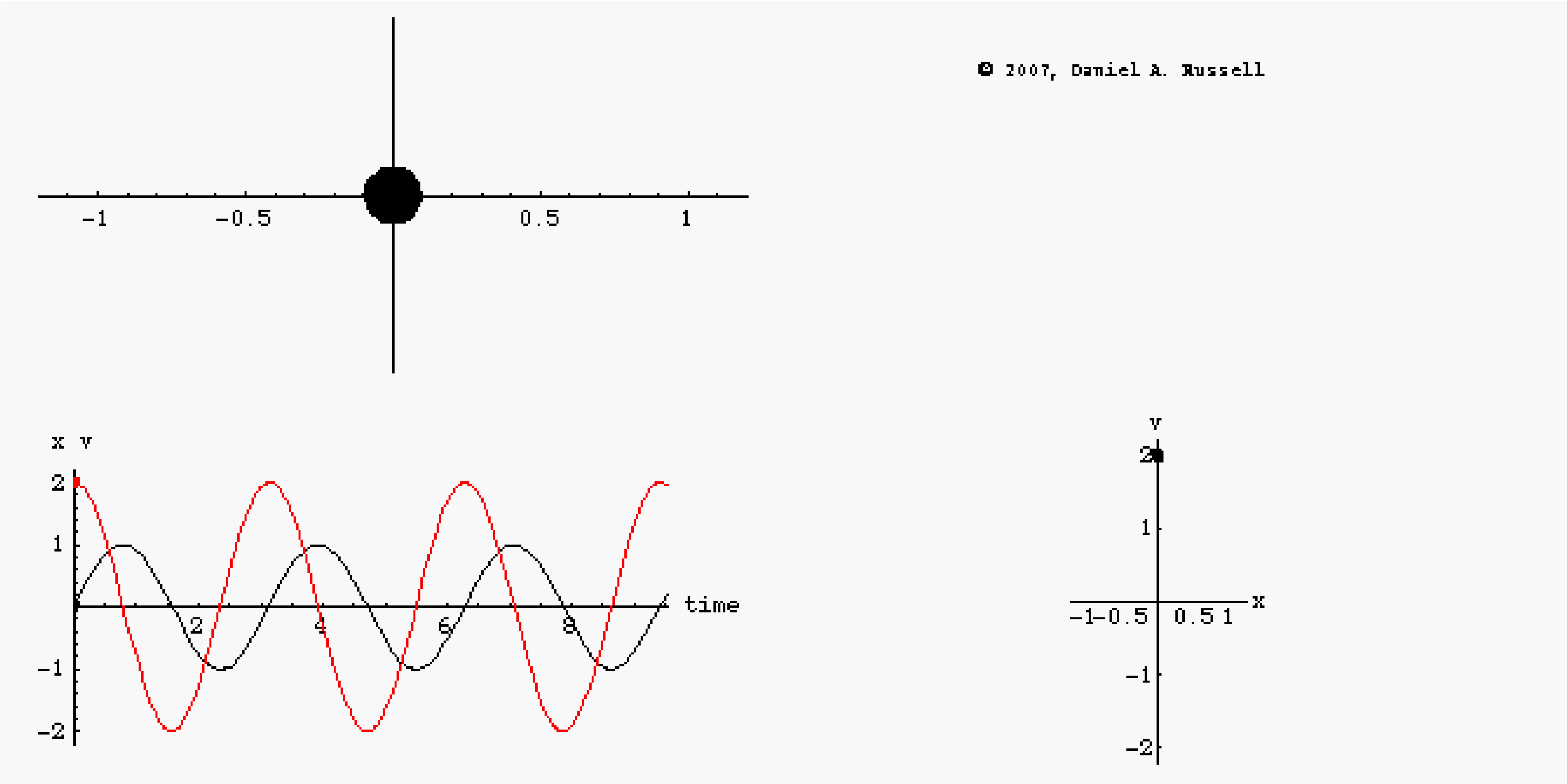
- The motion repeats itself, a consequence of the conservation of the total energy of the harmonic oscillator.

Due to $E_{total} = \frac{1}{2}kA^2$ and $\omega_0^2 = k/m$,

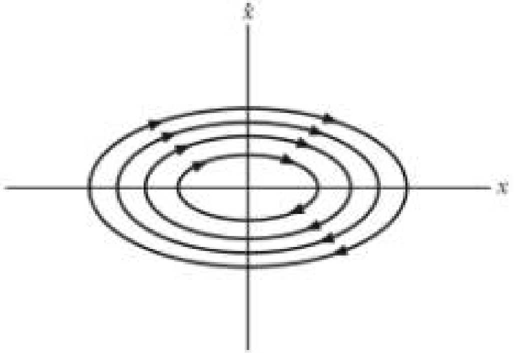
this leads to $\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1$

and finally $\frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = PE + KE = E_{total}$

Phase space diagram for undamped oscillator

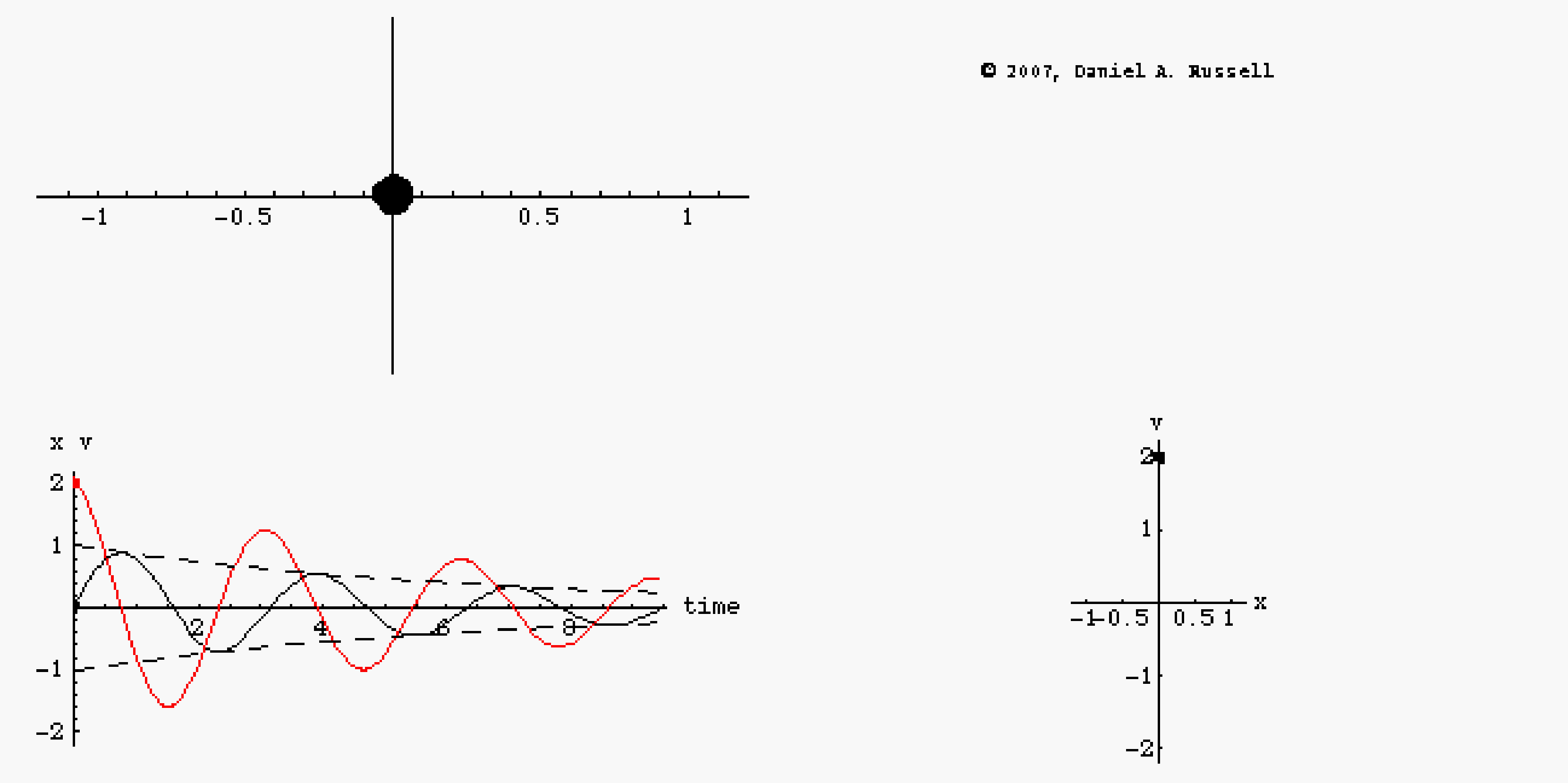


- An object undamped oscillating back and forth along x axis
- **Position x(t) (leads/lags)** **velocity v(t)** by $\ldots\frac{\pi}{2}\ldots$rad
- The phase diagram plot (phase space) is a combined plot of x(t) and v(t) giving a clockwise ellipse.



http://www.acs.psu.edu/drussell/Demos/phase-diagram/phase-diagram.html

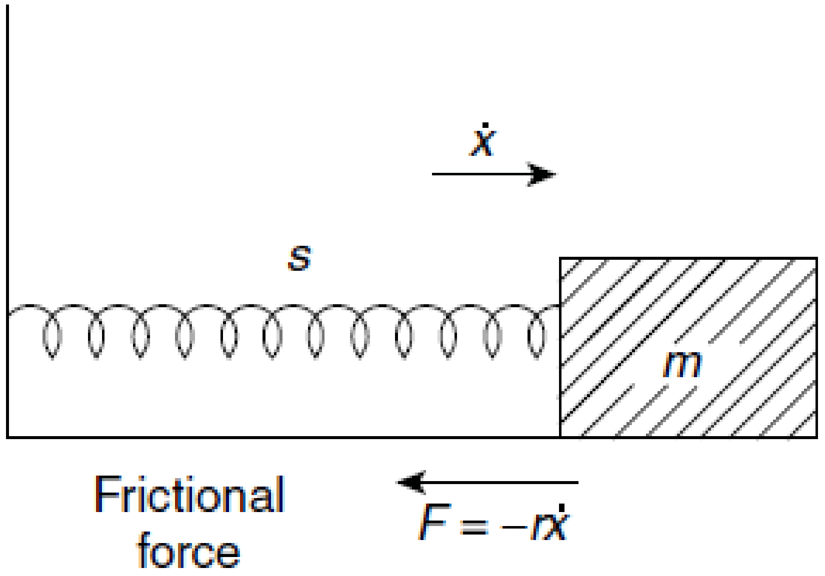
Phase space diagram for (lightly) damped oscillator



- The oscillator loses energy during each cycle.
- x(t) and v(t) exponentially decrease in amplitude as time proceeds.
- In classical mechanics, this phase space described as an "attractor" .

http://www.acs.psu.edu/drussell/Demos/phase-diagram/phase-diagram.html

Damped simple harmonic motion



- Equation of motion

$$m\ddot{x} + r\dot{x} + sx = 0$$

- General solution for the differential equation

$$x = C_1 e^{\frac{-r}{2m}t + t\sqrt{\frac{r^2}{4m^2} - \frac{s}{m}}} + C_2 e^{\frac{-r}{2m}t - t\sqrt{\frac{r^2}{4m^2} - \frac{s}{m}}}$$

- Where t = time and C_1, C_2 are arbitrary constants.

Three possible conditions

(1) Heavy damping : $r^2/4m^2 > s/m$

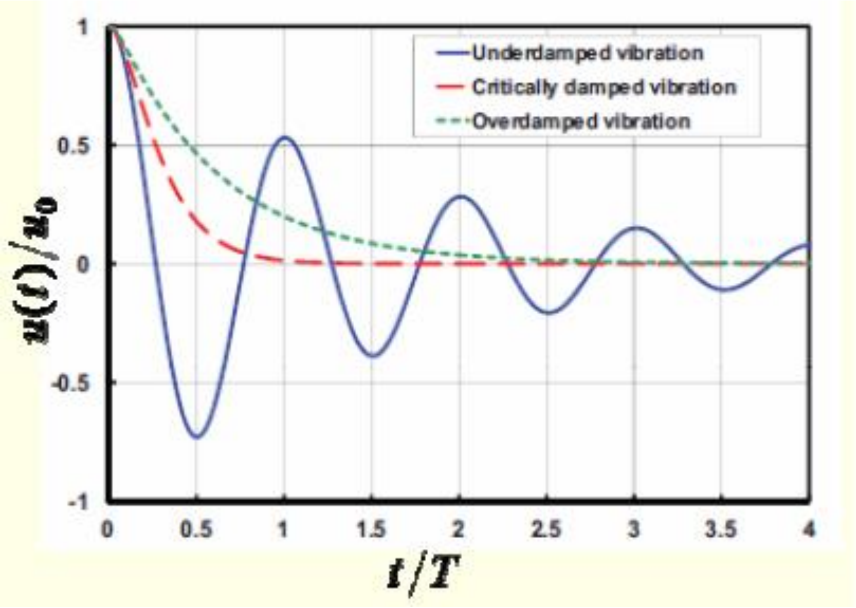
$$x = C_1 e^{-\frac{r}{2m}t + t\sqrt{\frac{r^2}{4m^2} - \frac{s}{m}}} + C_2 e^{-\frac{r}{2m}t - t\sqrt{\frac{r^2}{4m^2} - \frac{s}{m}}}$$

(2) Critically damping: $r^2/4m^2 = s/m$

$x =$ _ _ _ _ _ _ _ _ _ _ _ _

(3) Lightly damping: $r^2/4m^2 < s/m$

$x =$ _ _ _ _ _ _ _ _ _ _ _ _



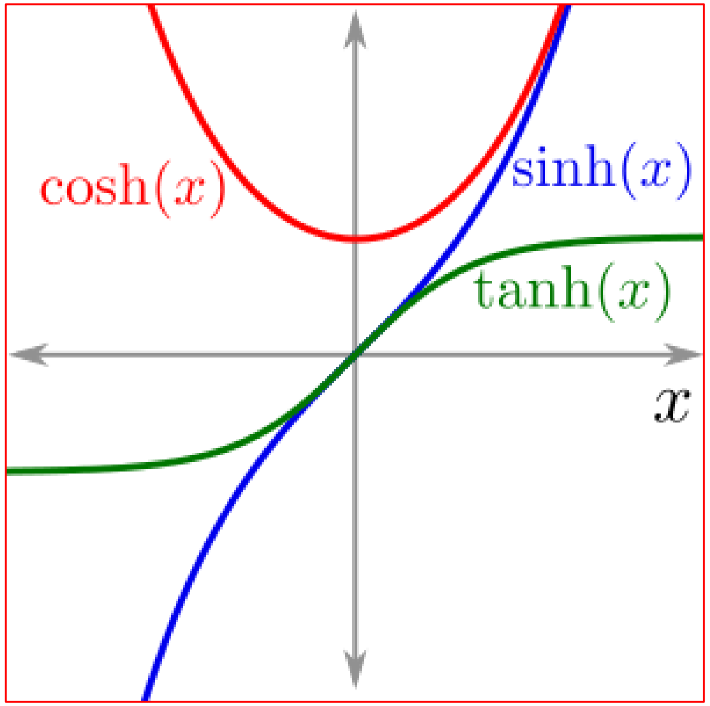
Displacement as function of time for one-dimensional damped oscillator

https://www.quora.com/What-is-damping-ratio

Heavy damping

- To illustrate the behavior of the heavy damping (over damping), the displacement x(t) can be rewritten as

$$x = e^{-pt}\left(F\cosh qt + G\sinh qt\right)$$



- Where $p = \frac{r}{2m}, q = \left(\frac{r^2}{4m^2} - \frac{s}{m}\right)^{\frac{1}{2}},$

$$F = C_1 + C_2, G = C_1 - C_2$$

- This represents non-oscillating behavior.

http://calculus.seas.upenn.edu/?n=Main.ComputingTaylorSeries

Derivation of the heavy damping displacement x

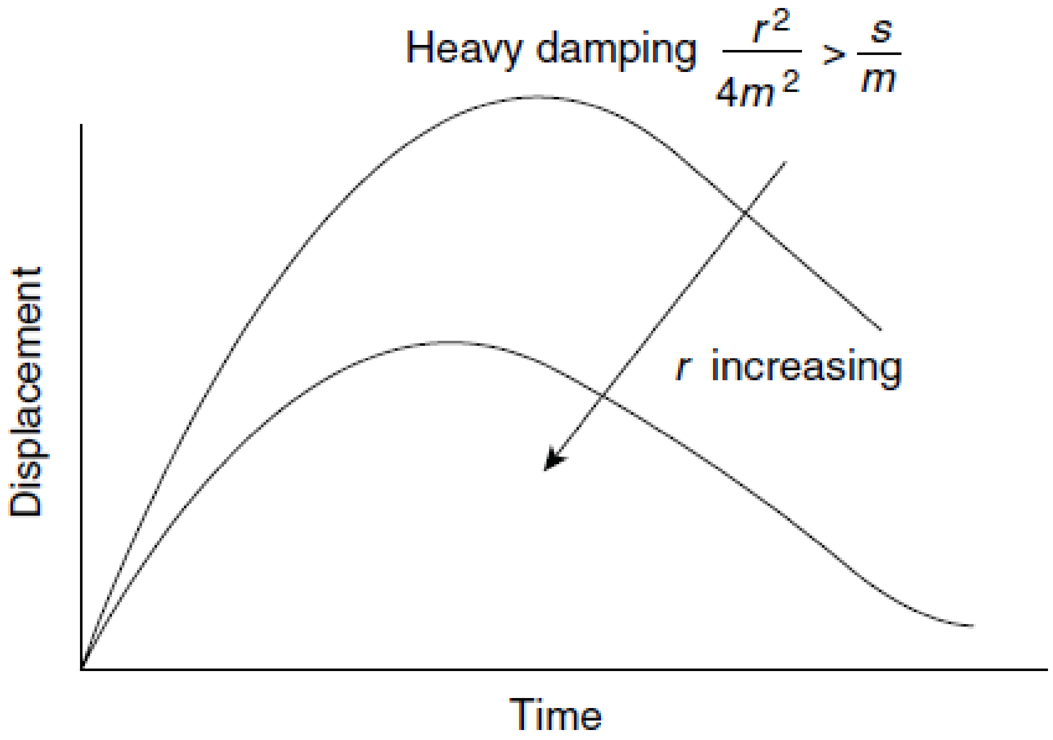
$$x = e^{-pt}\left(C_1 e^{qt} + C_2 e^{-qt}\right) = e^{-pt}\left(F\,\mathbf{cosh}\,qt + G\,\mathbf{sinh}\,qt\right)$$

$$= e^{-pt}\left(F\left(\frac{e^{qt}+e^{-qt}}{2}\right) + G\left(\frac{e^{qt}-e^{-qt}}{2}\right)\right)$$

$$= e^{-pt}\left(\left(\frac{F+G}{2}\right)e^{qt} + \left(\frac{F-G}{2}\right)e^{-qt}\right)$$

This gives $F = \left(C_1 + C_2\right)$ and $G = \left(C_1 - C_2\right)$.

Heavy damping graph



- Given an initial condition x(t=0) = 0.
- The displacement x(t) is written as

x(t) =

- The graph returns to zero displacement **quite slowly without oscillating** about its equilibrium position.

Critically damping

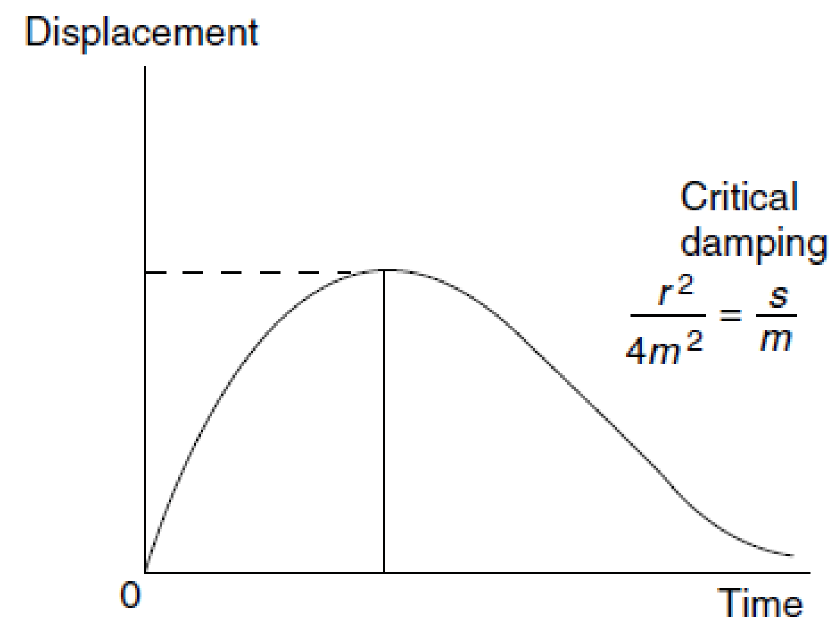
- Under the critically damping, the displacement becomes $x = (A + Bt)e^{-pt}$
- Where *A* is a **constant length** and *B* is a **given velocity** which depends on the boundary conditions.
- Suppose a critical damping system

has zero displacement at t = 0 and receives

an impulse which gives it an initial velocity V.

Determine the maximum displacement.

- Also the motion is non-oscillatory.



Comparison between heavy damping and critical damping

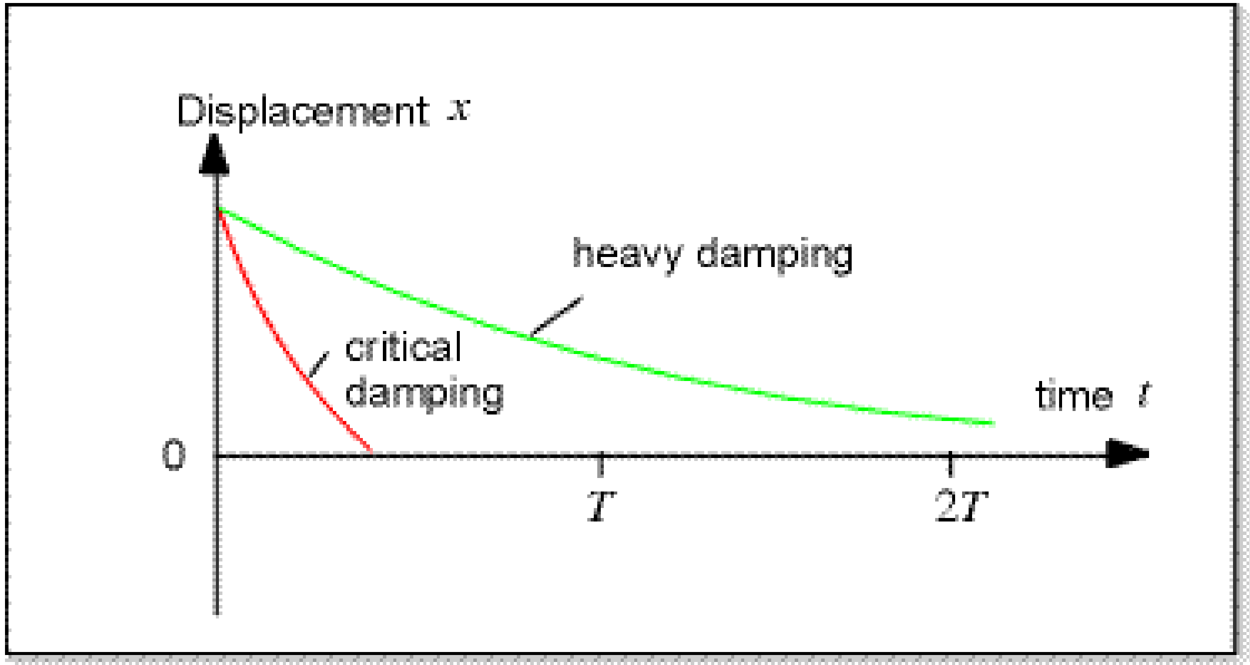
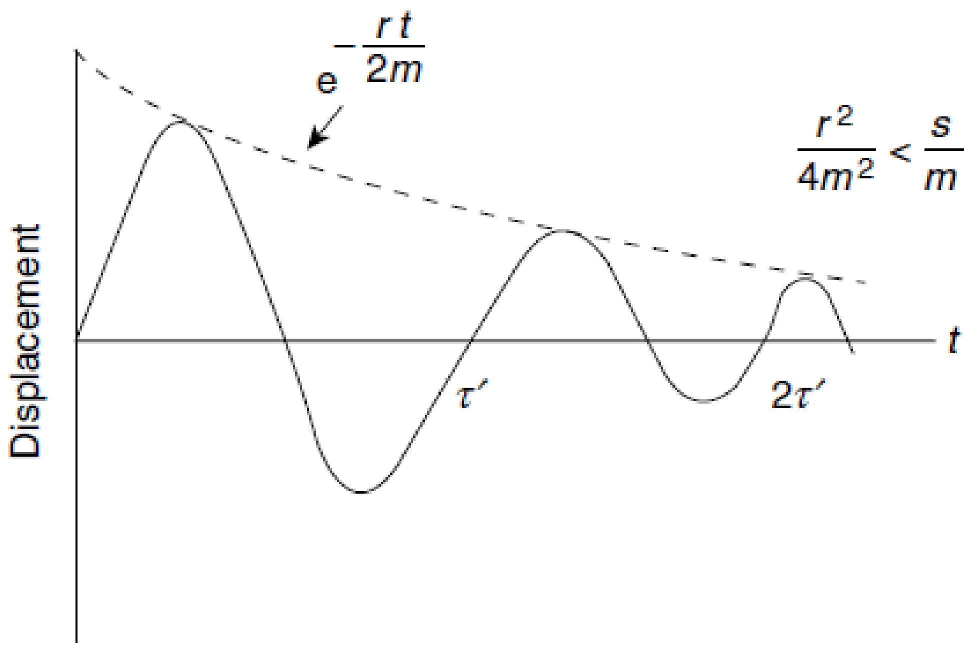


Fig.5.5.1

- Heavy damping of a damped oscillator will cause it to approach zero amplitude **more slowly** than for the case of critical damping.
- Critical damping is of practical importance in mechanical oscillators which experience sudden impulses and required to return to displacement in minimum time.

Lightly damping



- The behavior of the lightly damping oscillating system is described by

$$x = C_1 e^{-rt/2m + i\left(s/m - r^2/4m^2\right)^{\frac{1}{2}} t} + C_2 e^{-rt/2m - i\left(s/m - r^2/4m^2\right)^{\frac{1}{2}} t}$$

- The amplitude of the oscillator decays exponentially with time according to $\exp(-r/2m)t$
- **The damping coefficient *r* controls how fast the oscillating system comes to rest.**

Analysis of the lightly damping displacement

- Referring to $x = C_1 e^{-rt/2m + i\left(s/m - r^2/4m^2\right)^{\frac{1}{2}} t} + C_2 e^{-rt/2m - i\left(s/m - r^2/4m^2\right)^{\frac{1}{2}} t}$
- The bracket has the dimensions of frequency and can be written as $\left(s/m - r^2/4m^2\right)^{1/2} = \omega'$
- $\omega' < \omega; \omega^2 = s/m$ and **ω = frequency of ideal simple harmonic frequency (no damping!)**
- Therefore, $x = Ae^{-rt/2m} \sin(\omega' t + \phi)$; provided that $C_1 = \frac{A}{2i} e^{i\phi}, C_2 = -\frac{A}{2i} e^{-i\phi}$
- A and ϕ are constants which depend on the motion at t = 0.
- This lightly damping oscillation can then be compared to the simple harmonic oscillation.

Derivation of lightly damping displacement x

$$x = C_1 e^{-rt/2m + i\left(s/m - r^2/4m^2\right)^{\frac{1}{2}} t} + C_2 e^{-rt/2m - i\left(s/m - r^2/4m^2\right)^{\frac{1}{2}} t} = A e^{-rt/2m} \sin\left(\omega' t + \phi\right)$$

$$\text{if } \left(s/m - r^2/4m^2\right)^{1/2} = \omega'$$

this gives $C_1 e^{-rt/2m} e^{+i\omega' t} + C_2 e^{-rt/2m} e^{-i\omega' t} = \frac{Ae^{i\phi}}{2i} e^{-rt/2m} \left(e^{i\omega' t} - e^{-i\omega' t}\right)$

therefore, $C_1 = \frac{Ae^{i\phi}}{2i}$ and $C_2 = -\frac{Ae^{i\phi}}{2i}$

Differences between the undamped oscillation and underdamped oscillation

- Recall, the undamped oscillatory displacement : $x = A\sin(\omega t + \phi)$
- The underdamped oscillatory displacement : $x = Ae^{-rt/2m}\sin(\omega' t + \phi)$
- There are two difference : (1) the presence of the real exponential factor leading to a gradual death of oscillations and (2) the underdamped oscillator's angular frequency.
- The underdamped oscillator vibrates a little more slowly than does the undamped oscillator.
- **The period of underdamped oscillator is given by** $t' = \dfrac{2\pi}{\sqrt{\left(\dfrac{s}{m}\right)^2 - \left(\dfrac{r}{2m}\right)^2}} = \dfrac{2\pi}{\sqrt{\omega^2 - \left(\dfrac{r}{2m}\right)^2}}$

Energy Consideration

- The total energy of the damped harmonic oscillator is given by $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2$
- Differentiate the total energy with respect to t :

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + sx\dot{x} = \left(m\ddot{x} + sx\right)\dot{x}$$

- From the equation of motion of the damped oscillation : $m\ddot{x} + r\dot{x} + sx = 0$
- The time rate of change of total energy is found to be the product of the damping force and the velocity

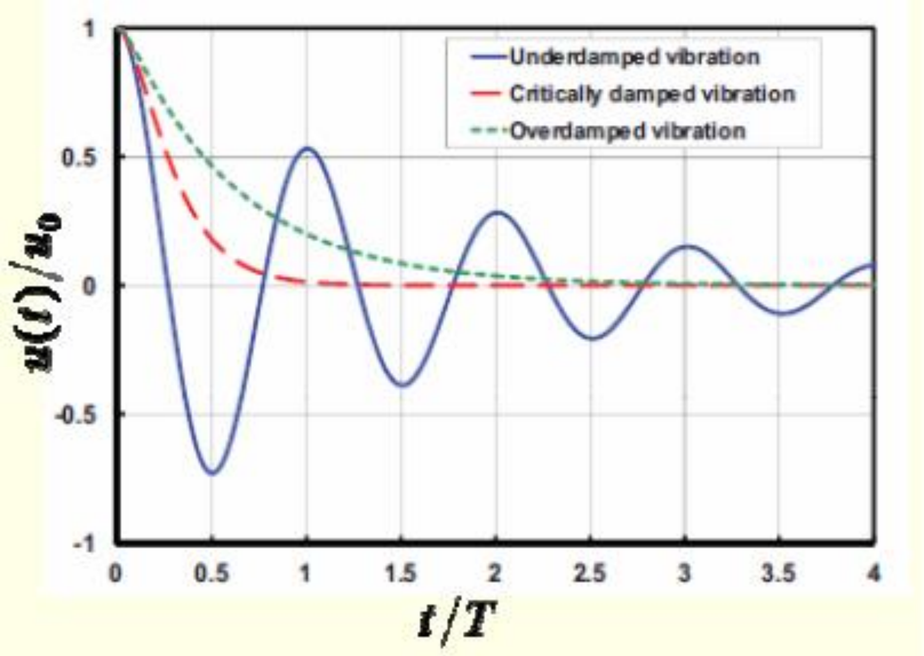
$$\frac{dE}{dt} = -r\dot{x}^2 = -\left(r\dot{x}\right)\left(\dot{x}\right)$$

- Because this always either zero (undamped) or negative (underdamped), **the total energy continually decreases**.

Damping ratio

$$\text{Damping Ratio} = \zeta = \frac{c}{c_{cr}}$$

- Damping Ratio (ζ = zeta) is dimensionless parameter which describes how an oscillating or vibrating body comes to rest.



$$\zeta = \frac{c}{c_{cr}} < 1.0$$ **Underdamped (structure oscillates to reach equilibrium)**

$$\zeta = \frac{c}{c_{cr}} = 1.0$$ **Critically Damped (structure does not oscillate to reach equilibrium)**

$$\zeta = \frac{c}{c_{cr}} > 1.0$$ **Overdamped (no oscillations and slower response to reach equilibrium)**

https://www.quora.com/What-is-damping-ratio

What is the damping ratio ζ?

- The damping ratio gives the level of damping in a system relative to critical damping.
- The damping ratio can be defined as ***the ratio of the damping coefficient in the system's differential equation to the critical damping coefficient***.

$$\text{Damping Ratio} = \zeta = \frac{c}{c_{cr}}$$

- **Example 1 : A spring-mass damper system has mass of 100 kg, stiffness of 3000 N/m and damping coefficient of 300 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Does the system oscillate?**

solution

Values for describing the damping of an oscillator

- Three different values are used to describe the behavior of a damped oscillator.
- They are composed of (1) **Logarithmic decrement**, (2) **Relaxation time** and (3) **Quality Factor or Q-Value**.

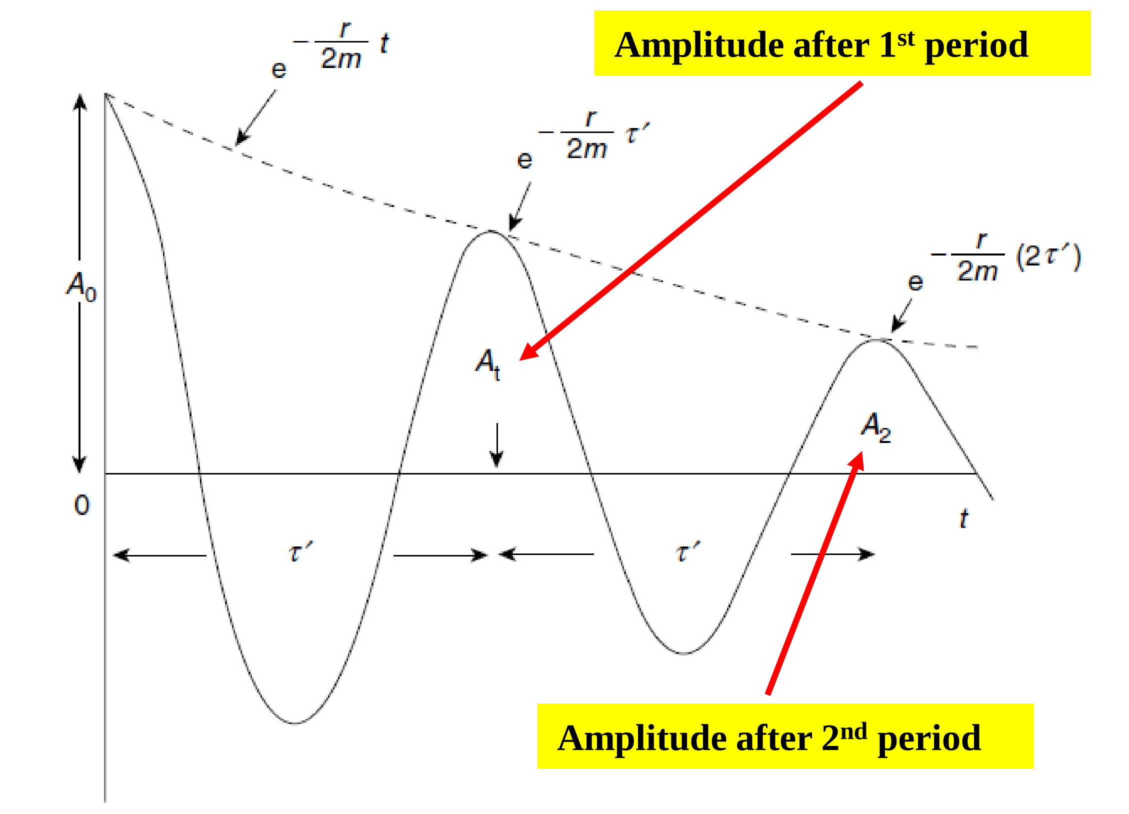
- Recall from the solution of a damped oscillation, the exponential decay factor **exp(-rt/2m)** expresses the rates at which the amplitude and energy is reduced.

Logarithmic decrement (1)

- **This value represents the rate at which the amplitude decreases with time.**
- Suppose the expression of the lightly damped oscillation, $x = Ae^{-rt/2m}\sin(\omega' t + \phi)$
- Given $\phi = \pi/2$ and at $t = 0$, $x = A_0$.
- The lightly damped can be described by

$$x = \boxed{}$$

- The behavior is shown as ------------->



Logarithmic decrement (2)

- From the graph in the previous slide, the amplitude at n^{th} period can be written as

$$A_n = A_0 e^{-rn\tau'/2m} \qquad ;\tau' = 2\pi/\omega'$$

- For example $A_1 = A_0 e^{-r\tau'/2m}, A_2 = A_0 e^{-r2\tau'/2m}, \ldots, A_n = A_0 e^{-rn\tau'/2m}$

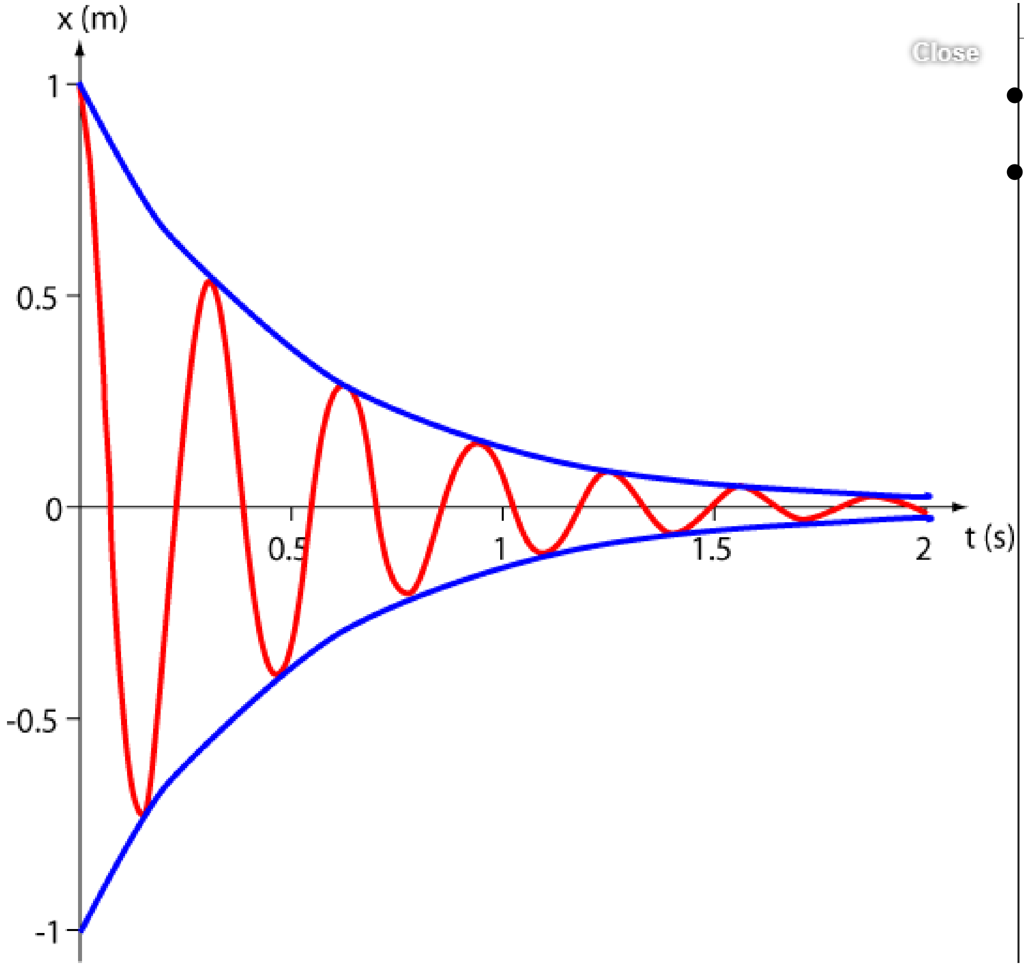
- Notice that $\dfrac{A_0}{A_1} = \dfrac{A_1}{A_2} = \dfrac{A_2}{A_3} = \ldots = \dfrac{A_{n-1}}{A_n} = \boxed{}$

- The logarithmic decrement $\delta = \ln \dfrac{A_{n-1}}{A_n} = \boxed{}$

Example 2

The frequency of a damped harmonic oscillator is one-half the frequency of the same oscillator with no damping. Find the ratio of the maxima of successive oscillations.

Example 3
Determination of logarithmic decrement



- Determine the logarithmic decrement δ from the graph.
- Given the relationship between the δ and the damping ratio ζ as follows,

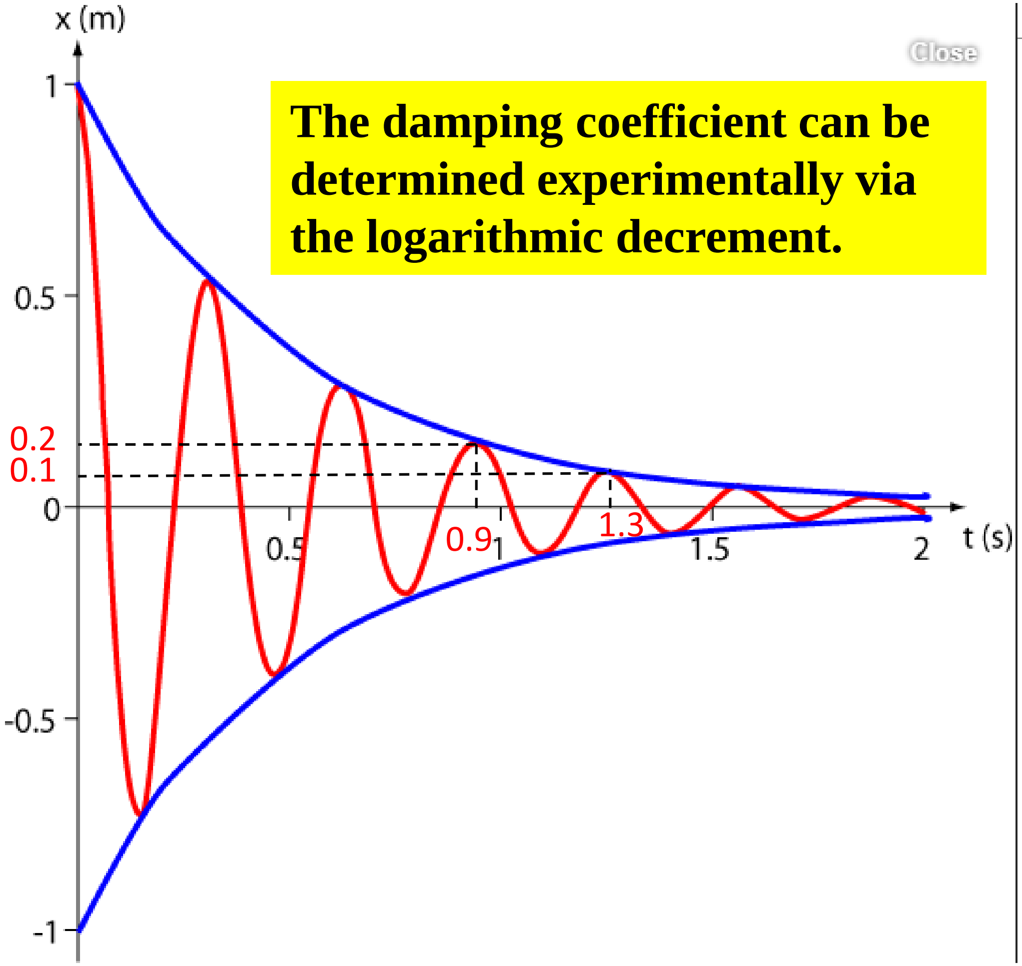
$$\zeta = \frac{1}{\sqrt{1 + (2\pi/\delta)^2}}$$

calculate the damping ratio and discuss the result.

https://www.quora.com/What-is-the-use-of-logarithmic-decrement

Solution

The damping coefficient can be determined experimentally via the logarithmic decrement.



You see a peak at $t \approx 0.9$, and $x(t = 0.9) = 0.2$. You see another peak at $t \approx 1.3$ with $x(t = 1.3) = 0.1$. You can now calculate the logarithmic decrement to be

$$\delta = \frac{1}{n} \log \frac{x(t)}{x(t+nT)}$$

with $n = 1$ representing the number of cycles, and $T = 0.4$ (from 1.3-0.9) which represents the period of oscillation. This gives

$$\delta = \log \frac{0.2}{0.1} = 0.69.$$

Using this you can now calculate the nondimensionalized Damping ratio of your system using

$$\zeta = \frac{1}{\sqrt{1+(2\pi/\delta)^2}} = 0.1$$

The value corresponds to the underdamping system.

Relaxation time

- This expresses the damping effect on the motion in terms of the time taken by **the amplitude to decrease by a factor** of ((1/*e*) = 0.368) of its original amplitude; *e* = 2.718.
- The relaxation time or modulus of decay is denoted as τ.
- From the expression $A_t = A_0 e^{-r\tau/2m} = A_0 e^{-1}; \therefore \tau = 2m/r$
- The relaxation time is a measure of **how rapidly the motion is damped out by friction**. The higher the value of *r*, the shorter the relaxation time.

Quality Factor or Q-Value (1)

- The reduction of the **total energy** of the damped oscillator depends on the exponential decay factor given by

$$E = E_0 e^{-rt/m}$$

- The time for the energy E to decay to $E_0 e^{-1}$ is given by $t = m/r$.
- During this time, the oscillator will have vibrated through $\omega' m/r$.
- The ratio is defined as the **quality factor** $Q = \omega' m/r$.
- This expresses the number of radians through which the damped system oscillates as its energy decays by a factor of $1/e$.

Quality Factor or Q-Value (2)

- The rate of energy loss is given as dE/dt where E = E_0exp(-rt/m).
- If a time interval corresponding to the energy decay is given by t' (equivalent to the period of oscillation), the energy loss becomes $-\Delta E = (dE/dt)t'$.
- Because $dE/dt = (-r/m)E_0 e^{-rt/m}$ and $Q = \omega' m / r$
- This leads to

$$\frac{\text{energy stored in system}}{\text{energy lost per cycle}} = \frac{E}{-\Delta E} = \frac{Q}{2\pi}$$

- In summary, Q is a measure of the rate at which an oscillator loses energy.

Analysis of the Q value

- The amplitude decay follows exp(-r/2m)t.
- Since energy $\propto$ amplitude squared, the decay term becomes exp(-r/m)t.
- The time taken for energy to change from $E_o$exp(-r/2m)t to $E_o$exp(-1) is found to be m/r.
- Since over one period τ' corresponding to 2π redians change,

  for m/r the radian change becomes $(2\pi/\tau')(m/r) = \omega' m/r$. This quantity is known as Q value.
- Now, if we consider the rate of energy change $dE/dt = (-r/m)E_o \exp(-r/2m)t = (-r/m)E$.
- If the time interval dt is chosen to be τ', dE becomes ΔE (energy loss per cycle) and E becomes a stored energy.
- Therefore, $E/-\Delta E=m/(r\tau')$. And due to $Q = \omega' m/r$, **$E/-\Delta E=Q/2\pi$**.
- The Q value expresses the ratio of the stored energy in an oscillator to the energy lost per cycle.

Example 4

The frequency of a damped simple harmonic oscillator is given by

$$\omega'^2 = \frac{s}{m} - \frac{r^2}{4m^2} = \omega_0^2 - \frac{r^2}{4m^2}$$

(a) If $\omega_0^2 - \omega'^2 = 10^{-6}\,\omega_0^2$, determine quality factor Q and logarithmic decrement δ.

(b) If $\omega_0 = 10^6$ and $m = 10^{-10}$ kg, determine the stiffness of the system and the resistive constant *r*.

(c) If the maximum displacement at t = 0 is 10^{-2} m. Determine the energy of the system and time taken to decay to 1/*e* of this value.

(d) Show that the energy loss in the first cycle is $2\pi \times 10^{-5}$ J.

Solution

(a) $\because \omega_0^2 - \omega'^2 = 10^{-6}\ \omega_0^2, \therefore m/r = 10^3/2\omega_0$. We also found that $\omega' \approx \omega_0$.

Therefore, quality factor $Q = \frac{\omega' m}{r} = \frac{\omega_0 m}{r} = 500.$

The logarithmic decrement $\delta = \frac{r}{2m}\tau' = \frac{r}{2m}\frac{2\pi}{\omega'} = \frac{r}{2m}\frac{2\pi}{\omega_0} = \frac{\pi}{500}$

(b) From $s = \omega_0^2/m$ = 100 N/m and $r = \frac{\delta}{2\pi} 2m\,\omega_0 = 2 \times 10^{-7}$ Ns/m

(c) Energy of the system at t = 0 : $\frac{1}{2} s A_0^2 = 5 \times 10^{-3}$ J

Time to decay by $1/e$: $t = m/r = 0.5$ ms

(d) From $\frac{Q}{2\pi} = \frac{E}{\Delta E}; \therefore \Delta E = 2\pi \times 10^{-5}$ J

Homework #2

1. Show that an overdamped oscillator can cross the equilibrium point one time at most.

2. Show that the damping ratio ζ is related to the logarithmic decrement via the following relationship:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$

3. Show that the quality factor of an electrical LCR series is $Q = \omega_0 L/R$ where $\omega_0 = 1/LC$